United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Maclaurin paper

Thursday 17 March 2022

England & Wales: Year 11
Scotland: S4
Northern Ireland: Year 12

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. You should write your solutions neatly on A4 paper. Staple your sheets together in the top left corner with the Cover Sheet on top and the questions in order.
5. Start each question on a fresh A4 sheet. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as π, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

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1. A fruit has a water content by weight of $m\%$. When left to dry in the sun, it loses $(m-5)\%$ of this water, leaving it with a water content by weight of 50%. What is the value of m?

2. (i) Expand and simplify $(x+1)(x^2-x+1)$.

(ii) Find all powers of 2 that are one more than a cube.

(A power of 2 is a number that can be written in the form 2^n, where n is an integer greater than or equal to 0.)

3. How many distinct triangles satisfy all the following properties:

(i) all three side-lengths are a whole number of centimetres in length;

(ii) at least one side is of length 10 cm;

(iii) at least one side-length is the (arithmetic) mean of the other two side-lengths?

4. A robot sits at the origin of a two-dimensional plane. Each second the robot chooses a direction, North or East, and at the sth second moves 2^{s-1} units in that direction. The total number of moves made by the robot is a multiple of 3. Show that, for each possible total number of moves, there are at least four different routes the robot can take such that the distance from the origin to the robot's final position is an integer.

5. The equation $x^2+bx+c=0$ has two different integer solutions, and the equation $x^2+bx-c=0$ also has two different integer solutions, where b and c are nonzero.

(i) Show that it is possible to find different positive integers p and q such that $2b^2=p^2+q^2$.

(ii) Show that it is possible to find different positive integers r and s such that $b^2=r^2+s^2$.

6. The diagram shows a circle with points A, B, C, D on its circumference and point E on chord BC. Given that $\angle BAC = \angle CED$ and $BC = 4 \times CE$, prove that $DB = 2 \times DE$.